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RENE DESCARTES*

BY

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Rene Descartes was born on March 31, 1596 at La Haye near Tours, the last child of his mother who died a few days after the delivery. To give an idea of the intellectual period in which he was born, the ages of his contemporaries then may be mentioned. Vieta 56, Napier 46, Harriot 36, Briggs 35, Shakespeare and Galileo 32 Kepler 25, Oughtred 21, Mersenne and Desargues 8 and 3 respectively, while Fermat (b. 1601), Roberval (b. 1602), Milton (b. 1608), Wallis (b. 1616), Pascal (b. 1623), Huyghens (b. 1629), Newton (b. 1642) and Leibnitz (b. 1646) were all born within the lifetime of Descartes. Rene's father provided his inquisitive son an excellent nurse who was often worried by him to substantiate her statements about heaven. These early fundamental doubts inspired Rene's life-work and destined him to become a great philosopher and a greater mathematician.

On account of his delicate health he was put to school late. His father wisely put him under the care of the good rector Father Charlet of the Jesuit College at La Fleche. Finding that the boy required more rest than the normal boys of his age, the good rector allowed him to lie in bed as late as he wished in the mornings, and to join his fellows in the class-room only when he felt like doing so. Rene was very grateful to the Father for this kindness and did much good work in silent meditations in bed. In his later days he recollected that the happiest period of his life were those quiet mornings in bed at La Fleche. In 1647, when he visited Pascal, another invalid mathematician of exceptional powers, he advised him not to allow himself to be disturbed in the mornings from his bed in order to preserve his health and mathematics. Pascal did not heed this advice and died early. At La Fleche Rene made some life-long friends. The most notable of them was Mersenne, later

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Father, the famous scientific amateur who made it his business to befriend and correspond with all the French mathematicians of his time and many foreign contemporaries also. Rene emerged from his college in his seventeenth year with all the equipment that the educational tradition of those days could give him.

His scepticism increased with his studies. He found the humanities that he studied were bereft of human significance and the logic of the schoolmen of the middle ages as barren as a mule for any creative human purpose. Disgusted with his studies, he resolved to see the world for himself, life in the raw. As he was financially well off to do as he pleased, he proceeded to Paris to enjoy life. He came into contact with the great mathematical thinkers of his day. He also gambled with enthusiasm and some success. The company of gamesters, however, soon tired him and he returned to his mathematics. But his hare-brained friends would not leave him. To get a little peace he went to war. He served the army in the Netherlands, Germany and Austria. But as there were no serious wars, the peaceful life of the camp also bored him though it gave him some leisure to dream about his mathematics and philosophy. His first idea of analytical geometry and philosophy came to him in some dreams that he had on the night of November 10, 1619, at Neuberg where his army was resting in its winter-quarters on the Danube. These dreams imply a sub-conscious working out of the possibilities of exploring natural phenomena by the aid of mathematics.

He took leave of his military activities in 1621. He returned to his father's home, sold away his portion of paternal property and invested the proceeds in such a manner as to fetch him an annual income of about five hundred pounds in modern currency. He then went away on a pilgrimage to Italy, and without being able to meet Galileo, the father of modern science, came back to Paris, the Paris of Cardinal Richelieu in 1625. Galileo's astronomical researches with his crude telescope had set all the natural philosophers of Europe grinding lenses. With his mathematical friends Mersenne and Mydorge, Rene became deeply interested in the theory of refraction of light and in investigating the best shape for glasses in optical instruments. As his hands were engaged in grinding glasses, his mind was equally engaged with reflections on the nature of man soul and God.

In spite of his lofty thoughts, Descartes had a weakness for gaudy and fashionable dress and went about in costly taffeta with

a broad-brimmed, ostrich-plumed hat. He had his own share of women in the world and had a daughter by one. The child's early death affected him deeply. He never married, and saved himself from designing Dutch widows by the announcement of his preference of truth to beauty.

The three years 1625–28 were the happiest of Descartes' life. But his increasing reputation brought him a horde of fashionable friends. He once more escaped his friends by taking military service, this time with the King of France at the siege of La Rochelle, where he met Cardinal Richelieu and each was impressed by the other. Till 1628, he had published nothing but on being acquainted with two good Jesuit cardinals who passionately loved the sciences and listened with admiration to his philosophy, he was too good a Catholic to resist their appeal to share his discoveries with the world, thereby avoiding the penalty of hell-fire which was threatened to be his lot otherwise.

The next twenty years of Descartes' life were spent in wandering all over Holland, methodically carrying on scientific and philosophic correspondence with all the leading intellects of Europe and occupying himself with all things that came in his way. Anatomy, Chemistry, Embryology, Medicine, Meteorology, Musical Theory, and Optics (in alphabetical order) were all grist to his mill. When he visited England he was introduced to the behaviour of the magnetic needle and forthwith magnetism was also included in his study. Theological speculations did not fail to attract his attention.

In 1634, Descartes was about to publish his physical theory of the Universe entitled Le Monde and was awaiting to see Galileo's latest book on science to put the final touches to his own. But instead of receiving a copy of the book he heard the shocking news that Galileo had been given up to the Inquisition, which forced him to abjure on his knees the Copernican doctrine that the earth moves round the sun. This is the tragedy of mathematical truths or for that matter any untimely truth which is opposed to vested interests. In the ecclesiastical imagination, Galileo's assertion of the earth's motion was tantamount to Galileo himself setting the earth in motion and bringing on some disastrous consequences like the earthquake. What business, indeed, had Galileo in prying into Nature's secrets and much more in publishing them? Nature herself was wroth with Galileo and connived in his punishment. Galileo's punishment proved such a deterrent to Descartes that he was afraid

to publish not only his Le Monde but his other projected works until some friends induced him to do so with sufficient guarantees of safety. In fact Cardinal Richelieu gave Descartes the privilege of publishing either in France or abroad anything he cared to write. But there were many Protestant theologians who savagely condemned Descartes' work as atheistic and dangerous to the State.

Descartes next devoted himself to a treatise on Universal Science. This was published in Leiden in 1637 under the title 'Discours de la Methode pour bien conduire sa raison et chercher la verite dans les sciences' accompanied by three appendices, La Dioptrique, Les Meteores, La Geometrie, of which the last proved to be of epoch-making importance. In 1641, he published his 'Meditationes' explaining at length the philosophy in his Discourse. The two books attracted the notice of the Princess Elizabeth (a grand-daughter of James I of England) who was a young prodigy of learning then living in her exile with her mother in the Hague, which unfortunately adjoined the little village where Descartes lived in his philosophic seclusion. An interview was arranged between the princess and the philosopher resulting in the latter ultimately consenting to be her tutor. Elizabeth took to Descartes' analytic methods with great enthusiasm. Even after she left Holland, she continued to be in touch with the philosopher until the day of his death.

In 1644 Descartes issued the Principia Philosophia the greater part of which was devoted to the physical sciences, especially the laws of motion and the theory of vortices. In the vortex-theory, universal motion is assumed to produce a number of whirlpools. The sun was supposed to be the centre of an immense vortex of matter in which the planets float and move round like straws in a whirlpool. Each planet again is the centre of a secondary vortex carrying the satellites and producing variations of density in the primary vortex causing elliptic instead of circular motion. The theory though crude and badly torn to pieces in the Principia of Newton yet deserves some credit as an attempt to describe the phenomena of the universe in terms of mechanical laws obtaining on the earth. This theory continued to be adopted only in France for about a century until in 1738 the Newtonian theory of gravitation ultimately superseded it mainly through the witty raillery of Voltaire.

In 1646 Descartes was having his quiet life in Egmond, meditating, gardening, and carrying on scientific correspondence with the intellectual lights of Europe. He was now fifty, world-famous and

indeed more famous than he ever cared to be. In 1647 he received a pension from the French Government in honour of his discoveries. But he was not left in peace to continue his great work. Queen Christine of Sweden had come to know of his existence. Like Elizabeth she had also a great passion for knowledge and wanted Descartes to be her instructor. Descartes was too snobbish to refuse the queen's request. The result was that he lost his peace and quiet morning hours of meditation in bed. Crawling out of bed at some ungodly hour in the dark, he had to climb into the carriage sent to 'collect' him and reach the queen's icy library punctually at 5 A.M. to give his lessons. He was also asked to plan out a Royal Swedish Academy of Sciences; he completed a few days before his death the statutes for this academy. Descartes could not get out of this work unless he used some rude words to his pupil, which he dared not. So he suffered until death got him release. He contracted inflammation of lungs, a disease caught from his host when nursing him. He died on February 11, 1650 aged 54. In 1667, the bones of Descartes were removed to France and re-entombed in what is now, the Pantheon. As Jacobi commented, it is often more convenient to possess the ashes of a man than the man himself. Descartes had an indiscreet temper which alienated him from his brilliant compeers and countrymen.

The collected works of Descartes first appeared in Latin at Amsterdam in nine volumes in 1690-1701. The French editions came much later. The latest edition by Charles Adam and Paul Tannery came out in nine volumes between 1897 and 1904.

Descartes was one of those universal personalities difficult to classify. He is claimed at the same time by philosophers, physicists and mathematicians as their man, perhaps with equal justification. He was one of the giants selected by Newton as his pedestal to enable him to see further. Descartes was one of those few men to whom it is given to renovate a whole department of human thought. Just as the age of physical science may be said to be inaugurated by Galileo, the age of modern mathematics may be said to begin with Descartes.

The modern algebraic notation is entirely due to him. He discovered some beautiful theorems in the Theory of Equations concerning the number and situations of the real roots, which are receiving their extensions even to-day. He gave a new solution of the equation of the fourth degree. It is interesting to remark that Bhaskara's solution (1150 A. D.) of a numerical equation of the

fourth degree suggests a method leading to a cubic which is no other than Descartes' auxiliary equation with a slight modification.. †

Descartes' greatest achievement lay in geometry. The synthetic methods of the Greeks did not offer any general tool for geometric research. Every problem was a jigsaw puzzle to be attacked by *ad hoc* special methods. Descartes' method introduced a new mathematical chakshus or vision with its parallel in simplicity and comprehensiveness only in the decimal notation which has pointed out a way of managing numbers in an infinitely ascending or descending progression. Changing the metaphor, we may say that geometry and algebra existed in different endogamic groups until Descartes celebrated the first marriage of the geometrical Eve and the algebraic Adam, himself acting as their first official priest, blessing their union with a public sermon. The result is 'analytical geometry'. 'The old-maidishness of Plato's rigidly corseted conception of what the Muse of Geometry should be,' disappeared with the Cartesian touch. Descartes not only released geometry from Platonic bondage but also gave her a pair of wings in the form of two co-ordinates with the help of which she can take flight into new worlds. The graphs which decorate the social sciences to-day are the direct descendants of this cartesian social reform.

The idea underlying analytical geometry is very simple. We all know that we can easily fix up a house if we know the name of the street in which the house is situated and the number of the house in the street. These two bits of information may be called the co-ordinates of the house. Even so, the position of a point in a plane may be uniquely fixed by two numbers, one giving the distance north or south (y) of a horizontal line in the plane, and the other the distance east or west (x) of a vertical line in the same plane. The point can now be represented by these two numbers which we call x, y co-ordinates. Suppose now the point begins to roam over the plane not in any random manner but according to a certain law, say, parallel to the horizontal line; then we say its y-co-ordinate is constant and write $y=k$. Again let it move such that its distance from the common point of the horizontal and vertical lines is constant; then we say its equation is $x^2+y^3=k^2$ while we know the path is a circle. Therefore $x^2+y^2=k^2$ represents a circle. Conversely from the equation we can pass on to the curve by finding x, y pairs satisfying the equation and locating them as points in a plane. If these points are sufficiently near they can be joined to obtain a curve. If we

† Vide *Current Science*, Vol. VII. No. 4 p. 179 (October, 1938.)

have two circles in the plane, we can write their corresponding equations, solve them and get number-pairs which tell us whether the circles cut each other, or not. Thus even without actually seeing the circles, we can work out their behaviour. Again, suppose a point chooses to move so as to keep its distance from a given point always equal to its distance from a given straight line, not passing through the point, what is its career like? The equation of the career can be readily written down by our new method and provides a comparison with that of the path of a projectile thrown in air. The similarity of the two equations shows that the two curves must be the same. The latter curve is known experimentally and therefore the former is laid bare. In this way, by a reduction of all curves to a common notation the solution of one problem would at once entail the solution of similar problems. In the language of Descartes the problem of curves is solved by a reduction to a problem of straight lines. In other words every geometric problem can be set to its algebraic expression and this implies that every improvement or discovery in Algebra is susceptible of application to geometry and vice-versa. When geometry and algebra thus get intimately inter-twined, it becomes impossible to improve the one without improving the other also. Henceforth we are released from the cobweb of lines in geometry and can write out equations more neatly and less confusingly than drawing lines which often give us misleading and false suggestions. This is another advantage in manipulating equations instead of manipulating lines. We would not be diverted by random configurations.

The above epoch-making method which I may call Descartes *monism* in mathematics, is elaborated in his La Geometrie an appendix of 116 pages to his Discours published in 1637. This work was made purposely obscure for fear that too much lucidity will make jealous people advertise it as trivial. It was appreciated more in the youthful universities of Holland than in the ecclesiastically controlled French universities. Van Schooten's Latin edition of La Geometrie as well as Wallis's treatise on Conic Sections popularised the analytic method among mathematicians. The result was a crop of discussions on various curves, their tangent and normal properties, their centre of gravity, and volumes of solids obtained by revolving curves. One of the important curves is the Cycloid, called the Helen of geometers on account of its beautiful properties and the controversies which their discovery engendered. It is the curve which forms beautiful arches. It is also the curve on which a body sliding from rest under gravity will reach the lowest point in the

same time from whatever point it starts. Descartes and Fermat succeeded in drawing tangents to this curve. As a challenge to Fermat, Descartes proposed his Folium for the construction of tangents. The correct shape of this curve was discovered in 1692 by C. Huyghens, the prince of philosophers in Holland, whose greatness Descartes had predicted. The famous quartic ovals invented by Descartes in connection with his inquiries about the best forms of converging lenses are now studied in connection with elliptic functions.

The analytical geometry was a worthy and a necessary precursor of the calculus of Newton and Leibnitz. Analytical geometry and Calculus are the two wonderful machines in which we have only to put certain functions or equations in order to make the result come out. It is no idle boast of Descartes that he had so far surpassed all geometry before him as Cicero's rhetoric surpassed the A, B, C. Descartes was not a great reader. Erudition was not his forte. Yet he was able to unfold a great science to the world, for he believed that the mind was not for the sake of knowledge but knowledge for the sake of the mind, and worked accordingly.

Ninety-nine years after Descartes had set the analytical ball rolling, Euler took it up for Mechanics. The glorious infection spread to Optics in 1827, when William Rowan Hamilton the greatest scientific man of Ireland constructed his algebraic theory of systems of rays, reducing all his researches to the story of a single function. Mathematical formulae seem to have an intelligence of their own showing the way as it were to their discoverers to seek hidden treasures in unknown directions in undreamt-of regions.

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ENUMERATION OF FINITE GROUPS

BY

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The three principal kinds of enumerations of groups of finite order which have been published are the permutation groups which can be represented on a given small number of letters, the abstract groups whose orders do not exceed a given small number, and thirdly those which separately contain a given small number of subgroups. In all of these cases the enumerations soon led to difficulties which made further progress along the particular line seem impractical, but the groups which have been enumerated throw much light on the nature of the general subject of group theory. These enumerations should therefore be regarded as convenient means for securing an insight into important properties of groups rather than as opening lines of investigation which should be pursued persistently along the direct lines to which they relate even if such progress seems to have no limit.

The enumeration of permutation groups was started at the beginning of the serious study of group theory as an autonomous subject of mathematics. Some of the earliest writers thereon including A. L. Cauchy (1789-1857) and E. Methieu (1835-1890), confined their attention in these enumerations to the possible orders of the permutation groups. This is due to the fact that in the applications to the solutions of algebraic equations it is especially interesting to know whether rational functions of n variables exist which have a given number of formal values when these variables are permuted in every possible manner. The fact that functions which have the same number of values may give rise to different equations of the same degree appeared at first to be of secondary importance. Most of the later writers considered however the determination also of all the possible groups of the same order, but even for the comparatively small degree 12 no one has as yet published all the possible intransitive permutation groups which exist.

The enumeration of the possible abstract groups of low orders has been carried somewhat further. In volume 175, page 252 (1936). of the *Journal fur die reine und angewandte Mathematik* the enumeration of all the abstract groups whose orders do not exceed 200 is

proposed as a prize problem. It is here stated that the solvable groups whose orders do not exceed 96 have been determined, while the insolvable groups can easily be determined even for a much higher limit of orders. On account of the much greater number of solvable groups than of insolvable groups for small numerical orders it is the former which have given the greater trouble in the published enumerations, which may be said to have been inaugurated by A. Cayley (1821-1895), who published occasional papers on the subject of group theory during about 40 years extending from 1854 to 1895 and contributed both to abstract groups and also to permutation groups.

The third of the three kinds of group enumerations to which we referred above is the most recent and appeared in various parts of the *Proceedings of the National Academy of Sciences*, beginning with volume 25, page 367 (1939) under the title "Groups which have a small number of subgroups". From the standpoint of abstract group theory this is the most natural type of enumerations since the earliest and most efficient method to study abstract groups is by means of their subgroups, and the theorem known as Lagranges theorem which affirms that the order of a group is divisible by the order of each of its subgroups may reasonably be regarded as the most fundamental theorem of the entire subject of group theory. It is therefore inaccurate to say that Sylow's theorem was the first to be proved among a set of closely related theorems on the structure of finite groups, as is stated in the *Encyclopedia Brittannica*, volume 10, page 914 (1938). Unfortunately a large number of other mathematical errors appear in this edition of this widely used encyclopedia. Cf. *Science*, December 1, 1939, page 512.

A group G which is not the identity contains at least two subgroups, viz., the identity and the group itself. These two subgroups are called *improper* subgroups of G while each of its other subgroups is called a *proper* subgroup of G, if the order of G exceeds 2. The identity is known as an actual subgroup of G so that G has always one more actual subgroups than proper subgroups, if it is not the identity. When G is the cyclic group of order p^m, p being an arbitrary prime number, then G contains exactly $m-1$ proper subgroups. It is not difficult to prove the following theorem relating to groups which contain a given number of subgroups: *A necessary and sufficient condition that there exists at least one cyclic group whose order is not a power of a prime number which contains k but no more than k proper subgroups, where k is an arbitrary positive integer, is that $k+2$ is not a prime number.*

The simplest case which comes under this theorem is when $k=2$. In this case the cyclic group of order pq, p and q being distance prime numbers, contains 2 but no more than 2 proper subgroups. The only other group which contains exactly 2 proper subgroups is the cyclic group of order p, p being an arbitrary prime number. When $k=4$ the cyclic group of order pq^2 contains exactly 4 proper subgroups for every pair of values of p and q, but in this case the non-cyclic group of order 9 also contains exactly 4 proper subgroups as well as the cyclic group of order p^5. These three groups are the only abelian groups which separately contain exactly 4 proper subgroups. There are also two non-abelian groups which separately contain exactly 4 proper subgroups, viz., the quaternion group and the non-cyclic group of order 6. It is easy to prove that every non-abelian group contains at least as many as 4 proper subgroups, and that the only non-abelian groups which contain exactly 4 proper subgroups are the two mentioned in the preceding sentence.

In view of the rapid recent changes in mathematical interests it may be desirable to add here that in 1898 there appeared the first part of a large mathematical encyclopedia entitled *Encyklopädie der Mathematischen Wissenschaften* which was completed in 1935. Even before it was completed it was generally recognized that at least the first volume thereof was already out of date and in need of many fundamental modifications in order to be abreast with current scholarship along its line. Since the summer of 1939 several parts of a revised volume 1 of this encyclopedia have been published, and it is interesting to observe how profoundly these parts, relating to algebra and the theory of numbers, differ from the corresponding parts of the first edition. In this connection it may be noted that much more space is devoted to group theory and its applications in this revised edition than in the first edition.

While only about 18 pages are devoted to finite discrete groups in this first edition, 51 pages are devoted to only a part of this subject in the revised edition under the heading of general group theory. It is also significant that the subject of abstract groups precedes that of permutation groups in the revised edition while the reverse order was followed in the first edition. Notwithstanding the rapid changes in mathematical interests and the many superficial mathematical activities of recent years group theory seems to have established its fundamental usefulness in the minds of a large number of the most successful mathematical investigators of modern times. It is also important to note the large increase in the number of the different writers on this subject in recent years.

A GENERALISATION OF THE CIRCULAR AND HYPERBOLIC FUNCTIONS

BY

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1·1 The object of this paper is to study a class of functions which include the hyperbolic and circular functions as particular cases.

1·2 Let $P_r(x)=\frac{1}{n}\sum_{s=0}^{n-1} t^{-rs}\exp(m_s x)$ $(r=0,\ 1,\ \cdots\ n-1)$, where $t=\exp(2\pi i/n)$ and $m_1,\ m_2\cdots$ are unequal and different from zero.

The functions $P_0(x),\ \cdots\ P_{n-1}(x)$ are linearly independent solutions of an ordinary linear differential equation of order n with constant coefficients, since the n-rowed determinant, whose element in the $(r+1)$th row and the $(s+1)$th column is t^{-rs} is not equal to zero.

1·3 It is easily seen that

$$\sum_{r=0}^{n-1} t^{sr}\,P_r(x)=\exp(m_s x)\quad (s=0,\ 1,\ 2,\ \cdots,\ n-1).$$

Multiplying together all these relations we get the circulantal identity

$$\begin{vmatrix} P_0, & P_1, & P_2, & \cdots\cdots, & P_{n-1} \\ P_{n-1}, & P_0, & P_1, & \cdots\cdots, & P_{n-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ P_1, & P_2, & \cdots & \cdots\cdots & P_0 \end{vmatrix} = \exp\left(\sum m_s x\right)$$

where P_r stands for $P_r(x)$. We note in particular that, when $\sum m_s=0$, the circulant is identically equal to 1.

1·4 $P_r(x)$ has the addition formula given by

$$P_r(x+y)=P_r(x)\,P_0(y)+P_{r-1}(x)\,P_1(y)+\cdots+P_0(x)\,P_r(y)+$$
$$+\,P_{n-1}(x)\,P_{r+1}(y)+P_{n-2}(x)\,P_{r+2}(y)+\cdots+P_{r+1}(x)\,P_{n-1}(y)$$

This may easily be demonstrated by replacing the P_s by their actual expressions.

1·5 It is seen from the identity

$$(P_0(x)+t\,P_1(x)+t^2\,P_2(x)+\cdots+t^{n-1}\,P_{n-1}(x))\times$$
$$(P_0(y)+t\,P_1(y)+\cdots+t^{n-1}\,P_{n-1}(y))$$
$$=P_0(x+y)+t\,P_1(x+y)+\cdots+t^{n-1}\,P_{n-1}(x+y),$$

and the addition formulæ that, if the left side of the identity be multiplied out making use of the simplification $t^{n+\lambda}=t^{\lambda}$, then the terms which are coefficients of t^r on both sides are equal. It follows that the expression for $P_r(x_1+x_2+\cdots+x_p)$ in terms of the $P_s(x_k)$ may be obtained by expanding the product $\prod_{k=1}^{p}\sum_{s=0}^{n-1} t^s\,P_s(x_k)$ making use of the simplification $t^{n+\lambda}=t^{\lambda}$ and collecting the terms which form the coefficient of t^r.

1·6 The geometrical significance of the determinantal identity is that the curve (in n-dimensional space) given by $x_r=A_r\,P_r(x)$ $(r=0,1,\cdots n-1)$, where the m_s in the P_s satisfy the condition $\sum_{s=0}^{n-1} m_s=0$, always lies on the algebraic manifold.

$$\begin{vmatrix} \frac{x_0}{A_0}, & \frac{x_1}{A_1}, & \cdots\cdots, & \frac{x_{n-1}}{A_{n-1}} \\ \frac{x_{n-1}}{A_{n-1}}, & \frac{x_0}{A_0}, & \cdots\cdots, & \frac{x_{n-2}}{A_{n-2}} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{x_1}{A_1}, & \frac{x_2}{A_2}, & \cdots\cdots, & \frac{x_0}{A_0} \end{vmatrix} = 1$$

1·7 $P_r(x)$ is not in general periodic. If $m_1, m_2, \ldots, m_{n-1}$ are all integral multiples, (+ve or −ve) of m_0, $P_r(x)$ is evidently periodic with period $2\pi i/m_0$. In the general case, corresponding to the periodicity relation

$$f(x+2\pi i/m_0)-f(x) = 0$$

of a function $f(x)$ with period $2\pi i/m_0$, we have a relation which may be stated as follows:

If $f(x) = A_0 e^{m_0 x} + A_1 e^{m_1 x} + \cdots + A_{n-1} e^{m_{n-1} x}$, then

$$f(x + \sum_n 2\pi i/m_r) - \sum_{n-1} f(x + \sum 2\pi i/m_r) + \cdots + (-1)^n f(x) = 0,$$

where $\sum_t 2\pi i/m_r$ denotes the sum of any t of the terms $2\pi i/m_r$ $(r = 0, 1, \ldots n-1)$, and $\sum f(x + \sum_t 2\pi i/m_r)$ denotes the sum of all the ${}_nC_t$ functions $f(x + \sum_t 2\pi i/m_r)$ corresponding to a given t. In case some of the m_s are multiples of another m the above relation may be replaced by a simpler one.

To prove this result we note that if $f(x + 2\pi i/m_0) - f(x) = f_1(x)$, say, then $f_1(x)$ is of the same form as $f(x)$ but does not contain the term $\exp(m_0 x)$ and similarly

$$f_1(x + 2\pi i/m_1) - f_1(x) = f_2(x), \text{ say,}$$

$$f_{n-2}(x + 2\pi i/m_{n-2}) - f_{n-2}(x) = f_{n-1}(x),$$

where $f_{n-1}(x)$ is periodic with period $2\pi i/m_{n-1}$, and so

$$f_{n-1}(x + 2\pi i/m_{n-1}) - f_{n-1}(x) = 0.$$

Eliminating $f_1, f_2, \ldots, f_{n-1}$, we get the required result.

2·1. Let us consider the following particular cases of the P-functions. Let $m_r = \rho^r$, and then denote $P_r(x)$ by $H_r(x)$. We find that

$$H_r(x) = \frac{x^r}{r!} + \frac{x^{n+r}}{(n+r)!} + \frac{x^{2n+r}}{(n+2r)!} + \cdots,$$

which is real for real values of x. The differential equation for $H_r(x)$ is readily seen to be $\frac{d^n y}{dx^n} = y$.

Also $\frac{d}{dx} H_r(x) = H_{r-1}(x), \ (r \neq 0)$

and $\frac{d}{dx} H_0(x) = H_{n-1}(x)$.

2·2. Let $m_r = e^{\frac{(2r+1)\pi i}{n}}$, and then let

$$P_r(x) = e^{r\pi i/n} C_r(x).$$

We have $C_r(x) = \frac{x^r}{r!} - \frac{x^{n+r}}{(n+r)!} + \frac{x^{2n+r}}{(2n+r)!} - \cdots,$

which is real for real values of x. The differential equation for $C_r(x)$ is

$$\frac{d^n y}{dx^n} + y = 0.$$

Also $$\frac{d}{dx}\, C_r\,(x) = C_{r-1}\,(x) \quad (r \neq 0)$$

and $$\frac{d}{dx}\, C_0\,(x) = -C_{n-1}\,(x).$$

The addition formula for $C_r\,(x)$ becomes

$$C_r\,(x+y) = C_r\,(x)\,C_0\,(y) + C_{r-1}\,(x)\,C_1\,(y) + \cdots + C_0\,(x)\,C_r\,(y)$$
$$- C_{n-1}\,(x)\,C_{r+1}\,(y) - \cdots - C_{r+1}\,(x)\,C_{n-1}\,(y),$$

and the circulantal identity takes the form

$$\begin{vmatrix} C_0, & C_1, & C_2, & \cdots\cdots & C_{n-1} \\ -C_{n-1}, & C_0, & C_1, & \cdots\cdots, & C_{n-2} \\ -C_{n-2}, & -C_{n-1}, & C_0, & \cdots\cdots, & C_{n-3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -C_1, & -C_2, & -C_3, & \cdots\cdots & C_0 \end{vmatrix} = 1,$$

where all the terms below the leading diagonal have the negative sign and the rest have the positive sign.

3·1 It could be shown that $P_r\,(x)$ has an infinite number of zeroes if $n \geqslant 2$.

We proceed to show that $C_r\,(x)$ has at least one positive zero for $n > 1$. We do not consider the case $n=2$, since the C-functions are then merely the cosine and sine of x. Let n be greater than or equal to 3. Then

$$\frac{C_r\,(x)}{x^r} = \frac{1}{r\,!} - \frac{x^n}{(n+r)\,!} + \frac{x^{2n}}{(2n+r)\,!} - \cdots\cdots$$

which tends to $1/r\,!$ as $x \to 0$.

Consider the quadratic in x^n, viz.,

$$\frac{x^{2n}}{(2n+r)\,!} - \frac{x^n}{(n+r)\,!} + \frac{1}{r\,!} = 0.$$

The roots of this equation are given by

$$2\,(n+r)\,!\;\; x^n = (2n+r)\,! \;\; \pm \sqrt{\{(2n+r)\,!\,\}^2 - \frac{4\,(2n+r)\,!\,\{(n+r)\,!\,\}^2}{r\,!}}$$

Consider the root x_1'', given by the negative sign. This is real and positive if

$$\frac{(2n+r)!}{(n+r)!} > 4\frac{(n+r)!}{r!} \qquad (r=0,\ 1,\ \cdots\ n-1)$$

i.e. if $\left(1+\frac{n}{r+1}\right)\left(1+\frac{n}{r+2}\right)\cdots\left(1+\frac{n}{r+n}\right) > 4.$

This inequality will be satisfied for $r=0,\ 1,\cdots n-1$ if it is satisfied for $r=n-1$. Now

$$\left(1+\frac{n}{n}\right)\left(1+\frac{n}{n+1}\right)\cdots\left(1+\frac{n}{2n-1}\right)$$

$$= 4\cdot\frac{(2n+1)}{(n+2)}\cdot\frac{(2n+3)(2n+4)\cdots(3n-1)}{(n+3)(n+4)\cdots\cdots(2n-1)} > 4.$$

Thus the quadratic in x^n has a positive root given by x_1''. Now

$$\frac{C_r(x)}{x^r} = \frac{1}{r} - \frac{x^n}{(n+r)!} + \frac{x^{2n}}{(2n+r)!} - x^{3n}\left\{\frac{1}{(3n+r)!} - \frac{x^n}{(4n+r)!}\right\}$$

$$- x^{5n}\left\{\qquad\right\} - \cdots\cdot$$

The expression within each of the flower brackets is positive for $x^n=(2n+r)!/2(n+r)!$ and so for $x=x_1$. Hence $C_r(x)/x^r$ is negative for $x=x_1$. Thus there is a root of $C_r(x)$ between 0 and x_1

Since $$\frac{C_r(x)}{x^r} = \frac{1}{r!} - \frac{x^n}{(r+n)!} + x^{2n}\left\{\frac{1}{(2n+r)!} - \frac{x^n}{(3n+r)!}\right\}$$

$$+ x^{4n}\left\{\qquad\right\} + \cdots\cdot,$$

and the expression within each of the flower brackets is negative for $x^n = \frac{(n+r)!}{r!}$, it follows that $\frac{C_r(x)}{x^r}$ is positive for $x^n = \frac{(n+r)!}{r!}$. Therefore there is a root of $C_r(x)$ between $\left\{\frac{(n+r)!}{r!}\right\}^{1/n}$ and x_1.

3·2 It is quite probable that there are an infinite number of real roots for $C_r(x)$. We can readily show that, when $n=4$,

$$C_0(x) = \prod_{n=1}^{\infty}\left\{1 - \frac{4x^4}{(2n-1)^4\pi^4}\right\}$$

and $$C_2(x) = \frac{x^2}{2}\prod_{n=1}^{\infty}\left\{1 - \frac{x^4}{4n^4\pi^4}\right\}.$$

It follows from this and the fact that $\frac{d}{dx} C_2(x) = C_1(x)$ and $\frac{d}{dx} C_0(x) = -C_3(x)$ that all the four functions $C_r(x)$ $(r=0, 1, 2, 3)$ have an infinite number of real roots.

4. The following particular cases and examples may be found interesting.

Let n be equal to 3. Then

$$H_0^3(x) + H_1^3(x) + H_2^3(x) - 3 H_0(x) H_1(x) H_2(x) = 1,$$
$$H_0(2x) = H_0^2(x) + 2H_1(x) H_2(x)$$
$$H_0(3x) = 1 + 9 H_0(x) H_1(x) H_2(x).$$
$$H_0(-x) = H_0^2(x) - H_1(x) H_2(x).$$

If $H_0(\lambda) = H_1(\mu) = H_2(\nu) = 0$, and $H_r(\lambda)$ is denoted by λ_r etc., then

$$H_0(x+\lambda) = \lambda_1 H_2(x) + \lambda_2 H_1(x),$$
$$H_0(3\lambda) = H_0(3\mu) = H_0(3\nu) = 1,$$
$$H_0(\lambda + \mu + \nu) = \lambda_1 \mu_2 \nu_0 + \lambda_2 \mu_0 \nu_1.$$

The periodicity law reduces to

$$H_r(x + 2\pi i) + H_r(x + 2\pi i\omega) + H_r(x + 2\pi i\omega^2)$$
$$= H_r(x - 2\pi i) + H_r(x - 2\pi i\omega) + H_r(x - 2\pi i\omega^2),$$

where $1, \omega, \omega^2$ are the cube roots of unity.

I thank Dr. R. Vaidyanathaswamy for his valuable guidance in the conduct of this investigation.

ON ABEL'S DIVERGENCE TEST FOR SERIES OF POSITIVE TERMS

BY

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1. Of the three theorems proved here, the first is a simple generalization of Abel's "test":

If $a_n \downarrow 0$ *as* $n \to \infty$, *the condition* $\lim n a_n = 0$ *is necessary for the convergence of* $\sum^{\infty} a_n$.

It is contained in the integral analogue of the test:

THEOREM A.—*If* $f(x) \downarrow 0$ *as* $x \to \infty$, *the condition* $\lim x f(x) = 0$ *is necessary for the convergence of* $\int^{\infty} f(x)\, dx$.

The second theorem is a generalization of certain results of Hardy and the last a deduction from one of Kronecker.

1·1. In what follows, the sequences $\{D_n\}$, $\{d_n\}$ are defined:

$$0 = D_0 < D_1 < D_2 < \cdots < D_n \to \infty,$$
$$d_n \equiv D_n - D_{n-1} \ (n \geqslant 1).$$

The starting point of our inquiry is

2. LEMMA 1 (KRONECKER). *The condition*

$$\lim \,(a_1 D_1 + a_2 D_2 + \cdots + a_n D_n)/D_n = 0$$

is necessary for the convergence of $\sum^{\infty} a_n$ (a_n *real or complex*). [1, H 10.]

2·1. If $a_n \downarrow 0$, we can take $D_n = \dfrac{1}{a_n}$ in the lemma and deduce therefrom Abel's test. More generally, taking

$$D_n = \frac{1}{(n \cdot l_1 n \; l_2 n \cdots l_k n)\; a_n},$$

where $l_1 n = \log n$, $l_2 n = \log \log n$ etc. and $n \geqslant m$ such that $l_{k+1} m > 0$, we infer:

If $(n \cdot l_1 n \cdot l_2 n \cdot \cdot \cdot \cdot l_k n)\ a_n \downarrow 0$, *the monotony asserting itself from a certain stage on, the condition* $\lim\ (n \cdot l_1 n \cdot l_2 n \cdot \cdot \cdot \cdot l_{k+1} n) a_n = 0$ *is necessary for the convergence of* $\sum^{\infty} a_n$. [Cp. 1, B 18]

2·2. The conclusion of the lemma does not imply $\lim \frac{a_n D_n}{d_n} = 0$ although $\lim \frac{a_n D_n}{d_n} = 0$ implies this conclusion. Even if we restrict ourselves to positive series, the condition $\lim \frac{a_n D_n}{d_n} = 0$ is not necessary for the convergence of $\sum a_n$, although the condition $\underline{\lim} \frac{a_n D_n}{d_n} = 0$ is. It is not difficult to construct examples of convergent series $\sum a_n\ (a_n > 0)$ for which there is a $\{D_n\}$ such that $\overline{\lim} \frac{a_n D_n}{d_n} = \infty$. Consider, for instance, the convergent series given by

$$a_n \begin{cases} = \dfrac{1}{\log n} \text{ when } n \text{ is of the form } 2^{m^2}, \\ = \dfrac{1}{n^2} \text{ otherwise.} \end{cases}$$

Taking $d_n = n^{-k}\ (0 < k < 1)$, we find $D_n \sim \frac{n^{1-k}}{1-k}$ so that

$$\frac{a_n\ D_n}{d_n} \sim \frac{1}{1-k}\ \frac{n}{\log n} \to \infty \text{ as } n = 2^{m^2} \to \infty.$$

Theorems 1 and 3 suggest a relation between $a_n\ (> 0)$ and d_n sufficient to ensure that the convergence of $\sum a_n$ shall imply $\lim \frac{a_n\ D_n}{d_n} = 0$.

3. THEOREM 1. *If* $\frac{a_n}{d_n} \downarrow 0$, *the condition* $\lim \frac{a_n\ D_n}{d_n} = 0$ *is necessary for the convergence of* $\sum^{\infty} a_n$.

Proof.—This may be deduced from Theorem A by taking

$$f(x) = \frac{a_n}{d_n} \text{ for } D_{n-1} < x \leqslant D_n.$$

COROLLARY 1. $\sum a_n \equiv \sum \frac{d_n}{D_n}$ *is divergent* (since $\frac{a_n\ D_n}{d_n} = 1$)

COROLLARY 2. The results in §2·1 can be deduced from Theorem 1 also by taking $d_n = 1, \frac{1}{n}, \frac{1}{n \cdot l_1 n} \cdot \cdot \cdot \cdot \cdot$ successively.

3·1. It is easy to prove that, if $d_n = O(1)$, the condition

$$\left.\begin{array}{l} \dfrac{1}{d_n} \log \dfrac{a_{n+1}/d_{n+1}}{a_n\, d_n} \\ \text{or } \dfrac{1}{d_n}\left(\dfrac{a_{n+1}\, d_{n+1}}{a_n/d_n} - 1\right) \end{array}\right\} = -\frac{\mu}{D_n} - O\left(\frac{1}{D_n{}^{\lambda}}\right) \qquad (\lambda > 1),$$

which is much narrower than the monotony of $\left\{\dfrac{a_n}{d_n}\right\}$, makes $\lim \dfrac{a_n D_n}{d_n} = 0$ *necessary and sufficient* for the convergence of $\sum a_n$ [3, 461-462.]

The last remark applies, for instance, to series such as

$$\sum a_{n+1} \equiv \sum d_{n+1} \frac{\prod\limits_{\nu=1}^{n} (\alpha d_\nu + D_\nu)(\beta d_\nu + D_\nu)}{\prod\limits_{\nu=1}^{n} (\gamma d_\nu + D_\nu)(\delta d_\nu + D_\nu)} \quad \left(\alpha,\ \beta,\ \gamma,\ \delta \text{ real};\ \gamma_1\delta \neq -\frac{D_1}{d_1},\ \frac{-D_2}{d_2},\ \cdots\right)$$

which reduce to the Hypergeometric type when $d_n = 1$.

4. THEOREM 2 (*a*). *If* $\dfrac{a_n}{d_n} \downarrow 0$, $\sum\limits^{\infty} a_n$ *is convergent, then* $\sum\limits_{1}^{\infty} D_n\left(\dfrac{a_n}{d_n} - \dfrac{a_{n+1}}{d_{n+1}}\right)$ *converges to the same sum as* $\sum\limits_{1}^{\infty} a_n$.

(*b*). *If, in addition,* $\left\{\dfrac{1}{d_n}\left(\dfrac{a_n}{d_n} - \dfrac{a_{n+1}}{d_{n+1}}\right)\right\}$ *is steadily decreasing, then* $\dfrac{a_n}{d_n} - \dfrac{a_{n+1}}{d_{n+1}} = o\left(\dfrac{d_n}{D_n{}^2}\right)$.

Proof. (*a*) Setting $b_n = D_n\left(\dfrac{a_n}{d_n} - \dfrac{a_{n+1}}{d_{n+1}}\right) \geqslant 0$, we find

$$b_n = \frac{a_n D_n}{d_n} - \frac{a_{n+1} D_{n+1}}{d_{n+1}} + a_{n+1}$$

and
$$\sum_{\nu=1}^{n} b_\nu = \sum_{\nu=1}^{n+1} a_\nu - \frac{a_{n+1} D_{n+1}}{d_{n+1}},$$

whence the first half of the theorem follows......

(*b*) The given condition

$$\frac{1}{d_n}\left(\frac{a_n}{d_n} - \frac{a_{n+1}}{d_{n+1}}\right) \geqslant \frac{1}{d_{n+1}}\left(\frac{a_{n+1}}{d_{n+1}} - \frac{a_{n+2}}{d_{n+2}}\right)$$

can be proved equivalent to

$$\frac{b_n}{d_n} + \frac{a_{n+1}}{d_{n+1}} \geqslant \frac{b_{n+1}}{d_{n+1}} + \frac{a_{n+2}}{d_{n+2}}$$

Now the convergence of $\sum a_n$ implies that of $\sum \frac{a_{n+1}}{d_{n+1}} d_n$ since $\frac{a_{n+1}}{d_{n+1}} d_n \leqslant a_n$

Hence $\sum a'_n \equiv \sum \left(b_n + \frac{a_{n+1}}{d_{n+1}} d_n \right)$ is convergent and $\left\{ \frac{a'_n}{d_n} \right\}$ is steadily decreasing.

Therefore, by Theorem 1,

$$\frac{b_n D_n}{d_n} < \frac{a'_n D_n}{d_n} \to 0.$$

COROLLARY 1. The case $d_n = 1$ [1, B 20] is due to Hardy.

With $d_n = 1/n \cdot l_1 n \cdot l_2 n \cdots l_k n$, we obtain

COROLLARY 2 (*a*). *If* $\sum^{\infty} a_n$ *is convergent and* $(n \cdot l_1 n \cdot l_2 n \cdots l_k n)\, a_n \downarrow 0$, *then*

$$\sum^{\infty} l_{k+1} n \left[n \cdot l_1 n \cdots l_k n \cdot a_n - (n+1) \cdot l_1 (n+1) \cdots l_k (n+1) \cdot a_{n+1} \right]$$

is also convergent.

(*b*) *If, in addition, the sequence*

$$\{ n \cdot l_1 n \cdots l_k n \left[n \cdot l_1 n \cdots l_k n \cdot a_n - (n+1) \cdot l_1 (n+1) \cdots l_k (n+1)\, a_{n+1} \right] \}$$

is steadily decreasing, then

$$n \cdot l_1 n \cdots l_k n \cdot a_n - (n+1) \cdot l_1 (n+1) \cdots l_k (n+1) \cdot a_{n+1} = o\,(1/n \cdot l_1 n \cdots l_k n \cdot l^2_{k+1} n)$$

4·1. Bromwich (*Infinite Series*, ed. 2, p. 430 ex. 5) observes that if $a_n > 0$ and $\left\{ \frac{a_n}{d_n} \right\}$ steadily decreases, the divergence of $\sum b_n \equiv \sum D_n \left(\frac{a_n}{d_n} - \frac{a_{n+1}}{d_{n+1}} \right)$ is secured by that of $\sum a_n$. If $\lim \frac{a_n}{d_n} = 0$, it is not difficult to prove this statement. If, however, $\lim \frac{a_n}{d_n} = \rho > 0$, the statement is not necessarily true as can be seen by taking $a_n = \frac{\rho}{n} + \frac{1}{n \cdot l_1^2 n}$, $d_n = \frac{1}{n}$, in which case $\sum b_n$ is convergent. Hence Theorem 2 (*a*) can be completed by

THEOREM 2 (*a'*). *If* $\frac{a_n}{d_n} \downarrow 0$, *then* $\sum^{\infty} a_n$ *and* $\sum^{\infty} D_n \left(\frac{a_n}{d_n} - \frac{a_{n+1}}{d_{n+1}} \right)$ *converge or diverge together.**

* The divergence part of Theorem 2 (*a'*) can be used to prove an important limit theorem of Cesáro (Bromwich, *loc. cit.*)

Corollary 2 (a) can be similarly completed.

5. The theorem in this section is established by a lemma which is an immediate deduction from a theorem of Minakshisundaram [2, Theorem 1 (ii)].

5·1. LEMMA 2. *If* (*i*) $\{\lambda_n\}$, $\{\mu_n\}$ *are the sequences*:

$$0=\lambda_0 < \lambda_1 < \cdots < \lambda_n \to \infty,$$

$$\mu_n = \lambda_n - \lambda_{n-1} \ (n \geqslant 1);$$

(*ii*) $\{S_n\}$ *is a real sequence such that* $S_{n-1} - S_n < O\left(\frac{\mu_n}{\lambda_n}\right)$;

(*iii*) $(\mu_2 S_1 + \mu_3 S_2 + \cdots + \mu_n S_{n-1})/\lambda_n \to 0$;

then $\overline{\lim}\ S_n = 0$

The reader will have no difficulty in supplying the proof.

5·2. THEOREM 3. *If* $\sum^{\infty} a_n$ $(a_n>0)$ *is convergent*,

$$\frac{a_n}{d_n} - \frac{a_{n+1}}{d_{n+1}} < O\left(\frac{d_n}{D_n^2}\right), \text{ then } \lim \frac{a_n D_n}{d_n} = 0.$$

Proof. The given condition

$$\frac{a_n}{d_n} - \frac{a_{n+1}}{d_{n+1}} < K \frac{d_n}{D_n^2} \quad (0 < K < \infty)$$

makes

$$\frac{a_n D_n}{d_n} - \frac{a_{n+1} D_{n+1}}{d_{n+1}} < \frac{a_n D_n}{d_n} - \frac{a_{n+1} D_n}{d_{n+1}}$$

$$< K \frac{d_n}{D_n}.$$

Hence if we replace λ_n, μ_n, S_n in Lemma 2 by D_n, d_n, $\frac{a_{n+1} D_{n+1}}{d_{n+1}}$ respectively, (ii) is satisfied.

In consequence of Kronecker's theorem, (iii) is also satisfied.

Lemma 2 therefore establishes that $\overline{\lim} \frac{a_n D_n}{d_n} = 0 = \underline{\lim} \frac{a_n D_n}{d_n}$, which is the result sought.

Remark. Theorem 3 gives us some idea of the gap between a condition (N) which makes $\lim \frac{a_n D_n}{d_n} = 0$ *necessary* for the convergence of $\sum^{\infty} a_n$ and a condition (N - S) which makes $\lim \frac{a_n D_n}{d_n} = 0$

necessary and sufficient for the convergence of $\sum a_n$. When $d_n = O(1)$, (N – S) in the form

$$\log \frac{a_{n+1}/d_{n+1}}{a_n/d_n} = -\mu \frac{d_n}{D_n} - O\left(\frac{d_n}{D_n{}^{\lambda}}\right) \qquad (\lambda > 1)$$

can be shown to involve

$$\frac{a_n}{d_n} - \frac{a_{n+1}}{d_{n+1}} < O\left(\frac{d_n}{D_n{}^{\mu+1}}\right) \qquad (\mu > 1).$$

On the other hand, the (N) in Theorem 3 is

$$\frac{a_n}{d_n} - \frac{a_{n+1}}{d_{n+1}} < O\left(\frac{d_n}{D_n{}^2}\right)$$

6. In conclusion, I state integral analogues of some of the results proved above, leaving their proofs to the reader.

THEOREM B. *The condition* $\lim\limits_{x\to\infty} \dfrac{\int^x t f(t)\,dt}{x} = 0$ *is necessary for the convergence of* $\int^\infty f(x)\,dx$.

THEOREM C (*a*). *If* $f(x) \downarrow 0$, *then* $\int^\infty f(x)\,dx$ *and* $\int^\infty -x f'(x)\,dx$ *converge or diverge together.*

(*b*) *If* $f(x) \downarrow 0$, $\int^\infty f(x)\,dx$ *converges*, $f''(x) \geqslant 0$, *then*

$$-f'(x) = o(1/x^2).$$

THEOREM D. *If* $\int^\infty f(x)\,dx$ $(f > 0)$ *is convergent,*

$$-f'(x) < O(1/x^2), \text{ then } \lim x f(x) = 0.$$

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COMPATIBILITY OF FISHER'S TESTS FOR INDEX NUMBER FORMULAE.*

BY

S. SUBRAMANIAN

Introduction

The problem of averaging the fluctuations in the prices or quantities (produced or consumed) of several commodities has been engaging the attention of statistical workers since long ago and various formulae have been put forward from time to time as suitable for the purpose. The unweighted arithmetic mean held the first place among these till recently with the simple geometric mean as a close second. These have now given place to the weighted arithmetic and geometric means. Further improvements have been suggested by Professor Irving Fisher in the form of the principle of 'crossing' and the application of the several tests for the soundness of a formula. These tests are seven in number and are called**

(1) the identity test;
(2) time reversal test;
(3) base test;
(4) circular test;
(5) commensurability test;
(6) determinateness test;
(7) factor reversal test.

If P_{ts} denotes the price index for year t with respect to the base year s and is given by $f\,(p_{1t},\ p_{1s},\ p_{2t},\ p_{2s}, \ldots\ldots q_{1t},\ q_{1s}, \ldots..)$ where the p's are the prices and the q's, the quantities, then test (1) requires that $P_{tt}=1$; test (2), $P_{ts}.\ P_{st}=1$; test (3) $\frac{\partial}{\partial s}\left(\frac{P_{ts}}{P_{t's}}\right)=0$; test (4), $P_{ts}/P_{t's}$ $=P_{tt'}$; test (5), $P_{ts}=f\,(\lambda_1\ p_{1t},\ \lambda_1\ p_{1s},\ \frac{q_{1t}}{\lambda_1}, \frac{q_{1s}}{\lambda_1}, \ldots\ldots\ldots\ldots)$; (6) P_{ts} should not become zero or infinite or indefinite when some price or quantity becomes zero; and (7), if p and q are interchanged in P_{ts} we should get Q_{ts} where $P_{ts}.\ Q_{ts}=\sum p_t\ q_t \div \sum p_s\ q_s$

* Paper read at the Conference of the Indian Mathematical Society (Lucknow, March 1938).

** R. Frisch: Journal of the American Statistical Association (1930).

It can be seen at once that if test (4) is satisfied, tests (1), (2) and (3) are automatically satisfied. Thus conditions (4), (5), (6) and (7) are the only independent ones. There is a widespread belief that some of these tests are incompatible with one another. Professor Frisch has observed* that tests (3), (5) and (6) cannot hold good at the same time. But as I have pointed out elsewhere† his conclusion is rendered incorrect by the fact that the solution of his differential equations is subject to certain limitations. The object of the present note is to show that the tests are perfectly compatible with one another and that a formula can be found so as to satisfy all of them. In part I of the note it is shown that a formula satisfying tests (4), (5) and (7) can also satisfy test (6); in part II some general remarks are offered concerning such a formula and its relation with Prof. Irving Fisher's ideal index.

Part I

1. The general form of P_{ts} satisfying tests (3) and (5) when no p or q vanishes, is given by

$$F(a_{1t}, a_{2t}, \ldots\ldots) \times G(a_{1s}, a_{2s}, \ldots) \cdot \left(\frac{p_{1t}}{p_{1s}}\right)^{c_1} \cdot \left(\frac{p_{2t}}{p_{2s}}\right)^{c_2} \ldots\ldots\ldots$$

where the a's are the values and F, G are arbitrary functions.‡

When, for instance, some p_t vanishes, the corresponding a is omitted from F and the corresponding price ratio is replaced by the inverse quantity ratio§.

Since test (4) is satisfied if the tests (3) and (1) are simultaneously satisfied, by applying test (1) to the form given above we get a formula satisfying tests (4) and (5). P_{ts} now takes the form

$$\frac{F(a_{1t}, a_{2t}, \ldots\ldots)}{F(a_{1s}, a_{2s}, \ldots\ldots)} \cdot \left(\frac{p_{1t}}{p_{1s}}\right)^{c_1} \cdot \left(\frac{p_{2t}}{p_{2s}}\right)^{c_2} \ldots\ldots$$

when no p or q vanishes. When some p_t is zero the corresponding a_t and a_s are to be omitted and the corresponding price ratio is to be replaced by the inverse quantity ratio.

2. If we now apply test (7) we get

$$P_{ts} = \left(\frac{p_{1t}}{p_{1s}}\right)^{c_1} \cdot \left(\frac{p_{2t}}{p_{2s}}\right)^{c_2} \ldots \times \sqrt{\frac{(\sum p_t\ q_t) \div \sum p_s\ q_s}{\left(\frac{a_{1t}}{a_{1s}}\right)^{c_1} \left(\frac{a_{2t}}{a_{2s}}\right)^{c_2}} \ldots\ldots} \quad \text{(A)}$$

* Frisch: *op. cit.*, p. 404.

† Subramanian: *Journal of the American Statistical Association* (1934).

‡ Frisch: *op. cit.*, p. 404.

§ Subramanian: *op. cit.*

It can be seen that if some p_t is zero, say p_{rt},

$$P_{ts} = \left(\frac{p_{1t}}{p_{1s}}\right)^{c_1} \cdot \left(\frac{p_{2t}}{p_{2s}}\right)^{c_2} \cdots \left(\frac{q_s}{q_{rt}}\right)^{c_r} \cdots \times \sqrt{\frac{\left(\sum p_t\ q_t\right) \div \sum p_s\ q_s}{\left(\frac{a_{1t}}{a_{1s}}\right)^{c_1} \cdots \left(\frac{a_{rs}}{a_{rt}}\right)^{c_r}}} \quad \cdots \quad \text{(B)}$$

It must be observed that all the c's are perfectly arbitrary constants in both (A) and (B). These constants can be so adjusted that P_{ts} takes a value other than zero even when some p or q vanishes. For instance in form (A) when p_{rt} is made zero the only factors that are affected are $(p_{rt}/p_{rs})^{c_r}$ in the numerator and $(a_{rt}/a_{rs})^{\frac{1}{2}c_r}$ in the denominator. These will give the result $k\,(p_{rt})^{\frac{1}{2}c_r}$* where k has a value other than zero. Now if we make c_r also zero, as we may, we see that P_{ts} does not vanish. Hence form (A) satisfies test (6). Form (B) can similarly be shown to satisfy test (6).

Thus forms (A) and (B) satisfy all the tests suggested by Professor Irving Fisher. It may be mentioned in this connection that it is unusual to provide, in a price index, for a commodity whose price is zero at any time. Therefore the determinateness test is not of much practical interest However, it is interesting to find that even when such a commodity is included, the index number can take a value other than zero. The surprising fact about forms (A) and (B) is that they are in geometric form and still satisfy the determinateness test.

Part II

A word may now be said about Professor Fisher's ideal index number. Since this index is essentially arithmetic in form it will not satisfy the circular test. The index takes the form $\sqrt{\frac{\sum p_t\ q_t}{\sum p_s\ q^s} \times \frac{\sum p_t\ q_s}{\sum p^s\ q^t}}$. It can be easily verified that form (A) in the present note can be written also as $\sqrt{\frac{\sum p_t\ q_t}{\sum p_s\ q_s} \cdot \frac{\pi\ (p_t\ q_s)^c}{\pi\ (p_s\ q_t)^c}}$. The essential difference between the two forms mentioned above is that in the first, the second factor is of an aggregative type which can also be

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* The practice nowadays is to weight each price ratio with the corresponding value. If $\frac{p_{rt}}{p_{rs}}$ were weighted with $p_{rt}\ q_{rt}$ the question reduces to finding the limit of x^x as $x \to 0$. We know $\underset{x\to 0}{\text{Lt}} \log x^x = \underset{x\to 0}{\text{Lt}}\ x \log x = \underset{x\to 0}{\text{Lt}} \frac{\log x}{1/x} = \underset{x\to 0}{\text{Lt}} - \frac{1}{x}\Big/\frac{1}{x^2}$ $= \underset{x\to 0}{\text{Lt}}$ of $x = 0$; thus $x^x \to 1$ as $x \to 0$.

If some p_s is zero, then $p_s\ q_s$ will be the proper weight.

considered as arithmetic whereas in the second form, the corresponding factor is of a geometric type. If the determinateness test is excluded, form (B) is rendered unnecessary and form (A) would do. As the latter now satisfies all the tests, it can very well be called the ideal index number. Of course, it is necessary to reach some agreed solution beforehand with regard to the arbitrary constants in the formula.

Conclusion

We have seen (i) that it is quite possible to find a formula satisfying all of Professor Irving Fisher's tests and that it is in the main of the *weighted* geometric type and (ii) that failure to pay proper attention to the weights has been responsible for the belief that some of Professor Fisher's tests are mutually incompatible.

ON THE FUNCTION DEFINED BY THE RELATION

$$\phi_r(x) = x\phi'_{r-1}(x)$$

BY

H. SREENIVASA SASTRY

1. The object of this paper is to find a general formula for the function $\phi_r(x)$ defined by the relation

$$\phi_r(x) = x\,\phi'_{r-1}(x), \qquad \ldots \quad (1)$$

in terms of $\phi_0(x)$ and its successive derivatives. We obtain by differentiation

$$\left.\begin{aligned} \phi_1(x) &= x\,\phi'_0(x) \\ \phi_2(x) &= x\,\phi'_0(x) + x^2\,\phi_0''(x) \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ \phi_r(x) &= a_{r,1}\,x\phi_0'(x) + a_{r,2}\,x^2\phi_0''(x) + \cdots + a_{r,r}\,x^r\phi_0^{(r)}(x) \end{aligned}\right\} \ldots (2)$$

where $a_{r,1}, a_{r,2}, \cdots, a_{r,r}$ are non-negative integers independent of x and $\phi_0(x)$ with

$$a_{r,1} = a_{r,r} = 1. \qquad \ldots \quad (3)$$

†1. To find a general formula for $a_{r,k}$, we note that $a_{r,k}$ is independent of $\phi_0(x)$, and hence we shall choose $\phi_0(x) = e^x$. From (2) we get

$$\sum_{n=1}^{\infty} \frac{n^r x^n}{n!} = \sum_{k=1}^{n} a_{r,k}\,x^k\,\phi_0^{(k)}(x) = \left(\sum_{k=1}^{r} a_{r,k}\,x^k\right) e^x,$$

which gives

$$\sum_{k=1}^{r} a_{r,k}\, x^k \equiv e^{-x} \sum_{n=1}^{\infty} \frac{n^r\, x^n}{n!}. \qquad \ldots \quad (4)$$

Equating coefficients of x^k $(1 \leqslant k \leqslant r)$ in (4) we get the relation

$$a_{r,k} = \frac{k^r}{k!} - \frac{(k-1)^r}{(k-1)!} + \frac{(k-2)^r}{(k-2)!\,2!} - \cdots + \frac{(-1)^{k-1}\cdot 1^r}{(k-1)!\,1!}$$

$$= \frac{\Delta^k\, k^r}{k!}. \qquad \ldots \quad (4')$$

where $\Delta f(k) = f(k) - f(k-1)$ and $\Delta^n = \Delta \cdot \Delta^{n-1}$. The above relation can also be written as*

$$a_{r,k} = \frac{k^{r-1}}{(k-1)!} - \frac{(k-1)^{r-1}}{(k-2)!} + \frac{(k-2)^{r-1}}{(k-2)!\,2!} - \cdots$$

$$= \frac{\Delta^{k-1}\, k^{r-1}}{(k-1)!}. \qquad \ldots \quad (4'')$$

2. By comparing $\phi_r(x)$ and $x\,\phi'_{r-1}(x)$ in (2) we get the recurrence relation

$$a_{r,k} = a_{r-1,k-1} + k\, a_{r-1,k}. \qquad \ldots \quad (5)$$

From which we deduce that

$$a_{r,k} = a_{r-1,k-1} + k\, a_{r-2,k-1} + \cdots + k^{r-k-1}\, a_{k,k-1} + k^{r-k}\, a_{k,k}. \quad \ldots \quad (6)$$

By using (3), (5) and (6) we can prove by double induction that

$$a_{r,k} = k^{r-k} + (k-1)\, \Delta k^{r-k} + \frac{k-2}{2!}\, \Delta^2 k^{r-k+1} + \cdots + \frac{\Delta^{k-1}\, k^{r-2}}{(k-1)!} \qquad \ldots \quad (7)$$

which gives another formula for $a_{r,k}$. The relation (7) can also be verified more easily by collecting the powers of $(k-p)$ which occur explicitly in (7) and showing that it is actually equal to that in (4).

2·1. Taking $k=r$ in (3′) and using (3) we get the well-known result

$$r^r - r\,(r-1)^r + \frac{r\,(r-1)}{2!}\,(r-2)^r - \cdots = r! \qquad \ldots \quad (8)$$

2·2. Taking $k=r-1$ in (7) we obtain

$$a_{r,r-1} = (r-1) + (r-2) + (r-3) + \cdots + 1$$

$$= \frac{r\,(r-1)}{2}. \ldots \qquad \ldots \quad (9)$$

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* I am indebted to Dr. C. N. Srinivasa Iyengar for the elegant relation (4) and for help in this investigation.

which, in conjunction with (4) for $a_{r+1,r}$ gives

$$r^r-(r-1)(r-1)^r+\frac{(r-1)(r-2)}{2!}(r-2)^r-\cdots=\tfrac{1}{2}(r+1)! \quad \ldots \quad (10)$$

2·3. From (5) we get successively

$$\begin{aligned} a_{r+2,r} &= a_{r+1,r-1}+r\ a_{r+1,r} \\ &= a_{r,r-2}+(r-1)\ a_{r,r-1}+r\ a_{r+1,r} \\ &\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot \\ &= a_{2,1}+2a_{3,2}+3a_{4,3}+\cdots+r\ a_{r+1,r} \end{aligned}$$

From which using (3) and (9) we get

$$a_{r+2,r}=\sum_{n=1}^{r}\frac{1}{2}n^2(n+1)=\frac{1}{24}r(r+1)(r+2)(3r+1).$$

But by (4)

$$a_{r+2,r}=\frac{\Delta^{r-1}r^{r+1}}{(r-1)!}$$

so that we get

$$r^{r+1}-(r-1)(r-1)^{r+1}+\frac{(r-1)(r-2)}{2!}(r-2)^{r+1}-\cdots$$
$$=\frac{1}{24}(3r+1)(r+2)!. \quad \ldots \quad (11)$$

We can get similar relations by considering $a_{r+3,r}$ etc. We may regard (10) and (11) as extensions of the known result (8).

3. THEOREM. *The integers $a_{r,2}$, $a_{r,3}$, $\cdots$ $a_{r,r-1}$ are all divisible by r when it is a prime number.*

For we have, for $2 \leqslant k \leqslant r-1$,

$$a_{r,k}=\frac{1}{(k-1)!}\left\{k^{r-1}-(k-1)(k-1)^{r-1}+\frac{(k-1)(k-2)}{2!}(k-2)^{r-1}-\cdots+(-1)^{k-1}\right\},$$

while

$$\begin{aligned} 0 &= (1-1)^{k-1} \\ &= 1-(k-1)+\frac{(k-1)(k-2)}{2!}-\cdots+(-1)^{k-1} \quad \ldots \quad (12) \end{aligned}$$

Multiplying (12) by $\frac{1}{(k-1)!}$ and subtracting from the expression for $a_{r,k}$ we get

$$a_{r,k}=\frac{1}{(k-1)!}\{(k^{r-1}-1)-(k-1)[(k-1)^{r-1}-1]+\cdots+(-1)^{k-2}[2^{r-1}-1]\}$$

From which we deduce that $a_{r,k}$ is divisible by r since by Fermat's theorem $p^{r-1}-1$ is divisible by r when r is a prime number and $p=2, 3, \ldots, r-1$.

3·1. Taking $\phi_0(x)=1+x+x^2+\cdots+x^n$ we get the formula $\phi_r(1)=1^r+2^r+\cdots+n^r$

$$=\sum_{k=1}^{r}\frac{1}{(k-1)!}\Delta^{k-1}k^{r-1}\phi_0^{(k)}(1),$$

where

$$\phi_0^{(k)}(1)=1\,2\cdots k+2\cdot3\cdot(k+1)+\cdots+(n-k+1)(n-k+2)\cdots n\cdot$$
$$=\frac{(n-k+1)(n-k+2)\cdots n\cdot(n+1)}{k+1}.$$

GLEANING

I hope I shall not create too many enemies if I flatly say that in mathematics classes, more often than in others, pupils are thought stupid by their teachers. In algebra and geometry one has to see relationships and act upon them quickly. But these are relationships of a special kind. A brilliant wit owes much to the ability to see unusual relationships. Some mathematicians have been brilliant wits. But not all.

T. H. PEAR: *The Art of Study*. p. 101.

(From A. A. KRISHNASWAMI IYENGAR, *Mysore*)

A Problem in Combinations *

The problem is:

Can m persons keep a watch on m days, n persons keeping a watch daily, so that any two persons go together only on two different days?

Evidently a solution can exist only if $m = \frac{n(n-1)}{2} + 1$.

I give below solutions for $n = 3, 4, 5$, and 6. It will be noticed that my solution for $n = 6$ is not elegant. I believe that a solution exists in the general case but I have not succeeded so far in obtaining it.

We denote the m persons by the numbers 1, 2, 3, ..., m.

(i) $n = 3$, $m = 4$.

The groups are :—

1, 2, 4;
2, 3, 1;
3, 4, 2;
4, 1, 3.

(ii) $n = 4$, $m = 7$.

The groups are :—

1, 2, 4, 7;
2, 3, 5, 1;
3, 4, 6, 2;
4, 5, 7, 3;
5, 6, 1, 4;
6, 7, 2, 5;
7, 1, 3, 6.

(iii) $n = 5$, $m = 11$.

The groups are :—

1, 2, 4, 7, 11;
2, 3, 5, 8, 1;
3, 4, 6, 9, 2;
4, 5, 7, 10, 3;
5, 6, 8, 11, 4;

* To me this problem was suggested by Dr. Chowla.

6,	7,	9,	1,	5;
7,	8,	10,	2,	6;
8,	9,	11,	3,	7;
9,	10,	1,	4,	8;
10,	11,	2,	5,	9;
11,	1,	3,	6,	10.

(iv) $n=6$, $m=16$.

The groups are :—

1,	2,	3,	4,	5,	6;
1,	2,	7,	8,	9,	10;
1,	3,	7,	11,	12,	13;
1,	4,	8,	11,	14,	15;
1,	5,	9,	12,	14,	16;
1,	6,	10,	13,	15,	16;
2,	3,	10,	12,	14,	15;
2,	4,	8,	12,	13,	16;
2,	5,	9,	11,	13,	15;
2,	6,	7,	11,	14,	16;
3,	4,	9,	10,	11,	16;
3,	5,	7,	8,	15,	16;
3,	6,	8,	9,	13,	14;
4,	5,	7,	10,	13,	14;
4,	6,	7,	9,	12,	15;
5,	6,	8,	10,	11,	12.

HANSRAJ GUPTA.

On the Converse of Fermat's Theorem

Fermat's theorem states that, if n is a prime and a is prime to n, then $a^{n-1}-1$ is a multiple of n. It is wellknown that the converse is not true. Mr. Kaprekar showed me a table of composite value of n for which $2^{n-1}-1$ is a multiple of n, in Sphinx. Let $F(x)$ denote the number of composite $n \leqslant x$, for which $2^{n-1}-1$ is a multiple of n. The object of this note is to prove

THEOREM II: $\underline{\lim}\ F(x)=\infty$.

To prove the above result, we want

THEOREM I: *Let*

(1) a be an integer > 1;

(2) p an odd prime;

(3) $P_1, P_2, \cdots\cdots, P_k$ odd prime factors of $a^{2p}-1$, different from the prime factors of a^2-1 and p; and

(4) Q be the product of any r number $P_1, P_2, \cdots\cdots, P_k$ Then $a^{Q-1}-1$ is a multiple of Q.

From a theorem of Euler, we know that $P_1, P_2, \cdots\cdots, P_k$ are of the form $2mp+1$. So $Q=2Mp+1$. Hence $a^{p-1}-1$ is a multiple of $a^{2p}-1$ and so a multiple of Q.

When $p \geqslant 5$, each of $2^{p-1}-1$ and $2^{p-1}+1$ should contain at least one prime of the form $2mp+1$. So corresponding to every $p \geqslant 5$, there should be at least one composite Q for which $2^{Q-1}-1$ should be a multiple of Q. Hence we get theorem II and

THEOREM III: $\varliminf F(x) \log\log x/\log x > 0$.

The general principle can be stated as

THEOREM IV: If Q be the product of any number of prime factors of a^m-1 such that $Q=Mm+1$, then Q is a factor of $a^{Q-1}-1$.

Though theorem IV is too general to be of any use, it shows that theorem I will not lead to the evaluation of all composite n for which $2^{n-1}-1$ is a multiple of n.

e.g. for theorem I: $3^{10}-1=8\times 11\times 11\times 61$.

Here, $a=3$, $p=5$, $k=3$, $P_1=11$, $P_2=11$ and $P_3=61$.

So $Q=11^2$, 11×61 or $11^2\times 61$. Hence $3^{120}-1$, $3^{670}-1$ and $3^{7380}-1$ are multiples of 121, 671 and 7381 respectively.

Annamalainagar S. S. PILLAI.

On the Sum of $\sum_{n=0}^{\infty} x^{12n}/(12n)!$

In a recent paper* Ludwig Silberstein has shown that

$$(1)\quad 1+\frac{x^3}{\underline{|3}}+\frac{x^6}{\underline{|6}}+\cdots\cdots=\frac{1}{3}\left\{e^x+2e^{-x/2}\cos\frac{\sqrt{3}}{2}x\right\}$$

$$(2)\quad 1-\frac{x^3}{\underline{|3}}+\frac{x^6}{\underline{|6}}-\cdots\cdots=\frac{1}{3}\left\{e^{-x}+2e^{x/2}\cos\frac{\sqrt{3}}{2}x\right\}$$

* *Phil Mag.* Vol. XXV page 1003 (1938).

$$(3)\quad 1+\frac{x^6}{\underline{|6}}+\frac{x^{12}}{\underline{|12}}+\cdots=\frac{1}{3}\left\{\cosh x+2\cosh\frac{x}{2}\cdot\cos\frac{\sqrt{3}}{2}x\right\}$$

We here sum two similar series.

The series (1), (2), (3) are evidently absolutely convergent for any finite x.

Replacing x by ix in (3), which is valid because of the convergence of (3) for all finite x (real or complex), we have since, $\cosh ix=\cos x$

$$(4)\quad 1-\frac{x^6}{\underline{|6}}+\frac{x^{12}}{\underline{|12}}-\cdots=\frac{1}{3}\left\{\cos x+2\cos\frac{x}{2}\cosh\frac{\sqrt{3}}{2}x\right\}$$

Taking half the sum of (3) and (4)

$$(5)\quad 1+\frac{x^{12}}{\underline{|12}}+\frac{x^{24}}{\underline{|24}}+\cdots=\frac{1}{6}\left\{\cosh x+\cos x+2\cosh\frac{x}{2}\cos\frac{\sqrt{3}}{2}x+2\cos\frac{x}{2}\cosh\frac{\sqrt{3}}{2}x\right\}$$

Benares Hindu University S. SASTRY.

SOLUTIONS TO QUESTIONS

Question 1700

(V. Thibault, Le Mans, France): With the digits 0, 1, 2, . . , 9 taken one at a time form a perfect square whose square root is of the form *abbba*.

Solution by D. K. Sen

This problem affords a good illustration of the methods of Practical Mathematics on which so much stress is nowadays laid in Europe and America during high school training. I shall give here a method of solution bearing on the principles of Practical Mathematics. In this method our aim is to obtain the solution in the simplest and shortest way. There will not be an academical discussion of the problem from the standpoint of Theory of Numbers or Analysis.

The problem is to find the square root of a number of ten digits wherein the figures 1, 2, 3, 4, 5, 6, 7. 8, 9 and 0 occur only once, the root being of the form *abbba*.

It is clear that a must be greater than 2, for otherwise the square of the number would consist of nine digits. This is also

evident if we take their logarithms; the logarithm of the square being clearly greater than 9, that of the root must be greater than 4·5 showing that the number is more than 30000.

Further the sum of the digits of the square is clearly 45; therefore it is divisible by 9. Hence its square root should be divisible by 3; or the sum of its digits must be a multiple of 3. Therefore a can only have the values 3, 6 or 9 so that $2a+3b$ may be divisible by 3. The rule of casting out of nines then shows that $2a+3b$ should be equal to 3 or 12 or 21 or 30 or 39, etc. Hence we get the following combinations only:

$$\left.\begin{matrix}a=3\\b=2\end{matrix}\right\}\left.\begin{matrix}a=6\\b=0\end{matrix}\right\}\left.\begin{matrix}a=6\\b=3\end{matrix}\right\}\left.\begin{matrix}a=6\\b=9\end{matrix}\right\}\left.\begin{matrix}a=9\\b=1\end{matrix}\right\}\left.\begin{matrix}a=9\\b=4\end{matrix}\right\}\left.\begin{matrix}a=9\\b=7\end{matrix}\right\}.$$

It is not very difficult to verify, by actual multiplication, that the last pair only gives the required square. Hence the number desired is 97779. As there are seven numbers to be tried the process should not be a prohibitive one. Even here one can shorten the labour of calculation. For example, let us take the fourth pair forming the number 69996 and try to write down a few steps of the multiplication similar to those of contracted multiplication, thus:

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**0016
------
    36
   108
  189
  ***
 ***
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There is no need to proceed further since we obtain in the sum (which is written out at the top instead of at the bottom for the sake of convenience) two 0's which is contrary to the given condition. The values of the various lines could be easily obtained from the following expression for the square in terms of a and b.

$$(a^2)+10.\ (2ab)+10^2.\ (2ab+b^2)+10^3.\ (2ab+2b^2)+10^4.\ (2a^2+3b^2)+\cdots$$

The values of the quantities in the brackets are written out in the successive lines.

Also solved by Messrs. V. K. Narasimhamurti, T. K. Raghavachari, K. Subba Rao and the proposer.

Question 1786

(K. S. K. Iyengar): Given a class $c(f)$ of functions $f(x)$ defined in $0 \leqslant x \leqslant 2\pi$ with the following properties:—

(1) f is L^2 in $(0, 2\pi)$.

(2) The Fourier series of f converges in the mean to f uniformly in the class $c(f)$.

(3) If f_1 and f_2 are any two functions of $c(f)$ then

$\int_0^{2\pi}(f_1 - f_2)^2\, dx \geqslant k$ where k is a positive constant;

prove that:

(A) 1. The class $c(f)$ is enumerable;

2. If the sequence $\{f_n\} \equiv c(f)$ and $R_n = \left\{\int_0^{2\pi} f_n^2\, dt\right\}^{\frac{1}{2}}$, then the index of convergence of R_n is finite;

(B) If the class c consists of trigonometric polynomials of degree not greater than n_0 and satisfy condition (3) given in the beginning then the index of convergence of the corresponding R_n's is not greater than $2n_0 + 1$;

(C) A Theorem similar to (A) and (B) can be enunciated with regard to algebraic polynomials, the index of convergence being not greater than $n_0 + 1$.

Solution by the Proposer

§ 1. Let $\{\phi_n\}$ be a complete normal orthogonal set over a finite interval say (0, 1) and let $\alpha_n = (f\phi_n) = \int_0^1 f\phi_n dx$ so that the series $\sum \alpha_n\phi_n$ converges in the mean to f.

We shall prove the results of the question for any complete normal orthogonal set taking the interval as (0, 1).

§ 2. *Preliminary remarks and a subsidiary Theorem*:—

The problem should be viewed as one in Hilbert space; there exists a 1 – 1 correspondence between functions of the class $c(f)$ and the set of points G(P) in Hilbert space with coordinates given by $P = (\alpha_1, \alpha_2 \cdots \alpha_n \cdots)$. The set of points G(P) being isolated (condition 3) and Hilbert space being separable, conclusion A(1) follows at once. However for the benefit of those not conversant with the terminology of Hilbert space, a direct proof of this result will be given later. Before proceeding to prove the main theorem, we shall first prove a subsidiary theorem which will be used in our arguments later.

THEOREM I: If G be a set of points in Euclidean p-space such that the distance PQ between any two points of G is $\geqslant k$, a fixed positive number; then

(1) G(P) is enumerable

(2) If $G \equiv \{P_n\}$ and O a fixed point, then the index of convergence of OP_n is not greater than p.

Proof: Because of the condition $PQ \geqslant k$, it is clear that in any finite p-sphere with O as centre, there will be only a finite number of points of G so that the points p of G can be so enumerated that

$$OP_1 \leqslant OP_2 \cdots \leqslant OP_n \longrightarrow \infty \text{ as } n \longrightarrow \infty \cdots \qquad (2\cdot1)$$

With $P_1, P_2 \cdots P_n$ as centres draw n spheres of radius $k/2$ then it is clear that none of these spheres overlap and they all lie in the sphere with centre O and radius $OP_n + k/2$, so that

$$n\left(\frac{k}{2}\right)^p \leqslant \left(OP_n + \frac{k}{2}\right)^p$$

$$\text{or} \qquad OP_n \geqslant \frac{k}{2}\left\{n^{1/p} - 1\right\} \cdots \qquad \ldots \quad (2\cdot2)$$

It is clear from this that $\sum OP_n^{-(p+a)}$ converges for $a>0$. Hence the theorem.

§ 3. *Proof of enumerability of $c(f)$*:—

Corresponding to any f of $c(f)$ we have the orthogonal series

$$\sum a_n^f \phi_n \sim f, \text{ and } \int_0^1 f^2\,dx = \sum_0^\infty (a_n^f)^2 \cdots \qquad (3\cdot1)$$

Given $\varepsilon > 0$, let $c_n(f)$ be the subclass of $c(f)$ such that for all f in it

$$\sum_{m=n+1}^{\infty} (a_m^f)^2 \leqslant \varepsilon \cdots \qquad \ldots \quad (3\cdot2)$$

Since $\sum\limits_1^\infty c_n(f) = c(f)$, the enumerability of $c(f)$ will follow from the enumerability of $c_n(f)$.

If f_1 and f_2 belong to $c_n(f)$, then

$$\int_0^1 (f_1 - f_2)^2\,dx = \sum_1^\infty (a_n^1 - a_n^2)^2 \geqslant k$$

$$\text{and } \left\{\sum_{n+1}^\infty (a_m^1 - a_m^2)^2\right\}^{1/2} \leqslant \left\{\sum_{n+1}^\infty (a_m^1)^2\right\}^{1/2} + \left\{\sum_{n+1}^\infty (a_m^2)^2\right\}^{1/2}$$

$$\leqslant 2\sqrt{\varepsilon}, \text{ by } (3\cdot2)$$

Hence $\sum_{n+1}^{\infty} (a_m^1 - a_m^2)^2 \leqslant 4\varepsilon$

Taking $\varepsilon = k/5$, we have

$$\sum_{1}^{n} (a_n^1 - a_n^2)^2 > \frac{k}{5} = k_1 \quad \cdot \cdot \cdot \qquad \ldots \quad (3\cdot3)$$

Consider now the set of points G (P) in Euclidean n-space whose co-ordinates are given by $(a_1, a_2 \cdot \cdot a_n)$ where $a_1, \cdot \cdot \cdot a_n$ are the first n orthogonal coefficients of any f in $c_n(f)$. Then because of (3·3), there is a 1 - 1 correspondence between G (P) and $c_n(f)$. Also, on account of (3·3) by Theorem I, G (P) is enumerable and hence $c_n(f)$ is enumerable.

§ 4. *Proof of A* (2):—

The Hypothesis (2) implies that, given any $\varepsilon > 0$ there must be an n_0 such that for all $n \geqslant n_0$ and all f in C (f)

$$\int_0^1 \left(f - \sum_1^n a_n^f \phi_n\right)^2 dx \leqslant \varepsilon$$

i.e. $$\sum_{n_0+1}^{\infty} \left(a_n^f\right)^2 \leqslant \varepsilon \qquad \ldots \quad (4\cdot1)$$

Using the terminology of §3, (4·1) implies

$$c(f) \equiv c_{n_0}(f)$$

Since there is a 1·1 correspondence between $C_{n_0}(f)$ and points P of G (P) with co-ordinates given by the 1st n_0 orthogonal coefficients of f in C (f) and since

$$PQ^2 = \sum_1^{n_0} \left(a_n^1 - a_n^2\right)^2 \geqslant k/5 \quad \left(\text{taking } \varepsilon = \frac{k}{5}\right),$$

by theorem I, $\sum \dfrac{1}{OP_f^{n_0+\alpha}}$ will be convergent for $\alpha > 0$ ... (4·2)

Since $\int_0^1 f^2\, dx \geqslant \sum_1^{n_0} (a_n^f)^2 = (OP_f)^2$,

A (2) follows at once ... (4·3)

[NOTE: Utmost one point P of G (P) may coincide with O because of condition (3) and in $\sum \dfrac{1}{(OP_f)^{n_0+\alpha}}$ that term may be omitted.

§ 5. *Proof of* (C) :—

If all f of $c(f)$ are defined by $f = \sum_{1}^{n_0} \alpha_n \phi_n$, $c(f)$ now corresponds to points in Euclidean n_0-space. Hypothesis (B) and theorem I imply as in (4·3) that $\sum 1/(R_f)^{n_0+\alpha}$ converges for $\alpha > 0$, where $R_f = \int_0^1 f^2 dx$.

... (5·1)

Taking for $\{\phi_n(x)\}$ the normal orthogonal algebraic polynomials in (0, 1), since by hypothesis the degree of f in C(f) is not greater than n_0, f can be expressed in terms of the first n_0+1 $\phi^1{}_n s$ and hence etc. Similar arguments will prove the conclusion about trigonometric polynomials.

§ 6. Although not connected with the problem directly, I have added some interesting applications of theorem I to a convergence problem :—

If $A_1, A_2 \cdot\cdot A_n$ be n independent vectors in Euclidean n-space (i.e. such that the determinant $\| A_1 A_2 \cdot\cdot A_n \| \neq 0$), and if $|N_1 A_1 + \cdot\cdot + N_n A_n|$ be the absolute value of the vector $N_1 A_1 + \cdot\cdot\cdot + N_n A_n$ then the multiple series

$$\sum_{N_n=-\infty}^{\infty} \cdot\cdot\cdot\cdot \sum_{N_1=-\infty}^{\infty} 1/|N_1 A_1 + \cdot\cdot\cdot + N_n A_n|^{n+\alpha}$$ is convergent for all $\alpha > 0$, where we omit $N_1 = N_2 = \cdot\cdot\cdot = N_n = 0$.

Proof: Let the end-point of the vector $N_1 A_1 + \cdot + N_n A_n$ be P and that of $N_1' A_1 + N_2' A_2 + \cdot\cdot\cdot$ be Q

Then $PQ = |(N_1 - N_1')A_1 + \cdot + (N_n - N_n')A_n| = |N_1'' A_1 + \cdot\cdot + N_n'' A_n|$ where all the $N''{}^1 s$ are not simultaneously zero.

It is easily proved that the hypothesis $\| A_1 \cdot\cdot A \|_n \neq 0$ implies that the minimum value of $|N_1 A_1 + \cdot\cdot + N_n A_n|$ is a positive number, where $N_1\ N_2 \cdot\cdot N_n$ can take all positive and negative integral values except $N_1 = N_2 = \cdot\cdot = N_n = 0$., so that

$PQ \geq \rho$ and by theorem A we get

$$\sum \cdot\cdot\cdot \sum 1/|N_1 A_1 + \cdot\cdot\cdot + N_n A_n|^{n+\alpha} \text{ converges for } \alpha > 0.$$

A particular case of this type of series with which Honours students are familiar is $\sum \sum 1/|m_1 \omega_1 + m_2 \omega_2|^s$ where the determinantal condition reduces to the usual form. imaginary part of $\omega_2/\omega_1 \neq 0$.

ANNOUNCEMENTS AND NEWS

1. The following gentlemen have been elected members of the Society :—

Som Datt Chopra Esq., M.A., Prof. of Mathematics, D. A. V. College, Hoshaiarpur.

P. L. Bhatnagar Esq., M.Sc., Lecturer, St. Stephen's College, Delhi.

2. The committee of the Indian Mathematical Society has been reconstituted as follows :—

President: Dr. R. Vaidyanathaswami, M.A., D.Sc., F.R.S.E., Reader, University of Madras.

Treasurer: Prof. L. N. Subramanian, M.A., Madras Christian College.

Librarian: Prof. R. P. Shintre, M.A., Fergusson College, Poona.

Other Members:

1. The Hon'ble Dr. Sir Shah Sulaiman, Kt., LL.D., D.Sc., Allahabad.
2. Dr. F. W. Levi, Dr. Phil., Nat., Calcutta.
3. Dr. A. Narasinga Rao, M.A., L.T., D.Sc., Annamalainagar.
4. Rai Saheb Dr. P. L. Srivastava, M.A., D.Phil., Allahabad.
5. Dr. A. N. Singh, D.Sc., Lucknow.
6. Prof. D. D. Kosambi, M.A., Poona.
7. Prof. S. Mahadevan, M.A., L.T., Madras.
8. Dr. M. R. Siddiqi, M.A., Ph.D., Hyderabad (Deccan).
9. Dr. Ram Behari, M.A., Ph.D., Delhi.

The committee of the Society has placed on record its appreciation of the services rendered by Dr. R. P. Paranjpye who has been President of the Society since 1936, and by Prof. M. V. Arunachala Sastri who has been a member of the committee for several years.

Prof. M. T. Naraniengar, M.A. formerly Professor of Mathematics at the Central College, Bangalore, died at Bangalore on the 9th October 1940. Prof. Naraniengar was the Foundation Editor of the *Journal of the Indian Mathematical Society* and laboured unceasingly and almost singlehanded for two decades to make it a worthy embodiment of the creative activity of India in the field of higher Mathematics. He was elected President of the Society for the period 1930-32 and was universally beloved and respected for his high sense of duty and the simplicity and saintliness of his character. A special number of the Mathematics Student will soon be published dedicated to his memory. The Indian Mathematical Society has passed a resolution expressing its deep sense of sorrow at his death and recording its appreciation of his services to the Society.

Dr. S. Sivasankaranarayana Pillai of the Annamalai University has been appointed Reader in Mathematics in the Travancore University and is expected to take up his new duties in January 1941.

Dr. N. S. Nagendranath, D.Sc. who was awarded the 1851 Exhibition Fellowship for work on Neuclear Physics at Cambridge, has returned to India and has been appointed Reader in the Andhra University.

QUESTIONS FOR SOLUTION

1793. (D. R. KAPREKAR): Let $\{a_n\}$ be a sequence defined by $a_n = a_{n-1} + 3n$, a_0 being any positive integer. Prove that, irrespective of the value of a_0, the number

$$a_0 + 10a_1 + 10^2 a_2 + \cdots 10^n a_n + \cdots$$

if continued sufficiently far will contain the recurring digits

004 115 226 337 448 559 670 781 893

or any cyclic permutation of the above digits.

1794. (P. KESAVA MENON): Find the value of

$$\sqrt[3]{\left[\frac{3}{2} + \frac{1\times 3}{2^2}\sqrt[3]{\left\{\frac{5}{2} + \frac{3\times 5}{2^2}\sqrt[3]{\left(\frac{7}{2} + \frac{5\times 7}{2^2} \times \cdots \text{to } \infty\right)}\right\}}\right]}$$

1795. (K. BALACHANDRAN)

If $\Delta(r) = \begin{vmatrix} 1 & 1 & 0 & 0 & \ldots & 0 \\ \frac{1}{3!} & \frac{1}{2!} & 1 & 0 & \ldots & 0 \\ \frac{1}{5!} & \frac{1}{4!} & \frac{1}{2!} & 1 & \ldots & 0 \\ \frac{1}{7!} & \frac{1}{6!} & \frac{1}{4!} & \frac{1}{2!} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \frac{1}{(2r-1)!} & \frac{1}{(2r-2)!} & \ldots & \ldots & \ldots & \frac{1}{2!} \end{vmatrix}$

and $(r > 1)$ show that $(2r-1)!\ \Delta(r)$ is an integer.

1796. (M. A. SRIRANGANARAYANA): Ox, Oy are two great circles of a sphere at right angles to each other. From any point P on the sphere perpendicular arcs PM, PN are drawn to Ox and Oy respectively. If $OM = x$, $ON = y$ be the coordinates of P, show that the equation to any great circle may be written in the form $A \tan x + B \tan y + C = 0$ and obtain the special forms

(i) $\cos\alpha \tan x + \sin\alpha \tan y = \tan p$

(ii) $\dfrac{\tan x}{\tan a} + \dfrac{\tan y}{\tan b} = 1$

(iii) $\tan y = \sqrt{(m^2 - \tan^2 c)}\, \tan x + \tan c$

1797. (A. A. KRISHNASWAMI AYYANGAR): Two conics S, S′ have vanishing θ, θ'. From any point on their real common chord, a tangent is drawn to meet S in A and a secant to cut S′ in B, C. Shew that S′ and the conic through A, B, C having three-pointic contact with S at A are such that two of their common tangents meet on a common chord.

GLEANING

e in all its glory

What a quantity of abstruse science for a bit of string! Let us not be surprised. A pellet of shot swinging at the end of a thread, a drop of dew trickling down a straw, a splash of water rippling under the kisses of the air, a mere trifle, after all, requires a titanic scaffolding, when we wish to examine it with the eye of calculation. We need the club of Hercules to crush a fly.

Our methods of mathematical investigation are certainly ingenious; we cannot too much admire the mighty brains that have invented them; but how slow and laborious they appear when compared with the smallest actualities! Will it never be given to probe reality in a simpler fashion? Will our intelligence be able one day to dispense with the heavy arsenal of formula? Why not?

Here we have the abracadabric number *e* reappearing, inscribed on a spi 'er's thread. Let us examine, on a misty morning the meshwork that has been constructed during the night. Owing to their hygrometrical nature, the sticky threads are laden with tiny drops, and, bending under the burden, have become so many catenaries, so many chaplets arranged in exquisite order and following the curve of a swing. If the sun pierce the mist, the whole lights up with iridescent fires and becomes a resplendent cluster of diamonds. The number *e* is in its glory.

Geometry, that is to say, the science of harmony in space, presides over everything. We find it in the arrangement of the scales of a fir-cone, as in the arrangement of an Epeira's lime-snare; we find in the spiral of a snail-shell, in the chaplet of a spider's thread, as in the orbit of a planet; it is everywhere, as perfect in the world of atoms as in the world of immensities.

And this universal geometry tells us of an Universal Geometrician, whose divine compasses have measured all things. . . . . . . I prefer that, as an explanation of the logarithmic curve of the Ammonite and the Epeira, to the worm screwing up the tip of its tail. It may not perhaps be in accordance with latterday teaching, but it takes a loftier flight.

HENRI FABRE,
from L. G. Simon's *Fabre and Mathematics and other Essays.*